SOLUTION OF CERTAIN PROBLEMS OF HYDROMECHANICS OF FILTRATION OF A HOMOGENEOUS LIQUID IN A THIN LAYER WITH MACROCRACKS

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Solution of a number of problems of the hydromechanics of macrocracks in a thin layer during the filtration of a homogeneous liquid is presented. A method of solving these problems using the trigonometric series is given.

A system of functional equations for the filtration flow in a thin layer containing macrocracks was obtained in [1, 2]. The basic hypothesis used in [1, 2] was, that in sufficiently narrow cracks the filtration flow is laminar and obeys the Darcy law. From the assumption of conservation of flow near the macrocrack element, the boundary condition for the velocity potential in and outside the cracks was found [1]. In constructing the velocity potential a major role is played by complex integrals with the Cauchy or Hilbert type kernels. The simplest converse problems were studied and it was shown [1, 2] that, if the discharge function $\omega(s)$ is prescribed, the cracks with either blunt or sharp ends exist.

Paper [3] dealt with the direct problem for the case when the crack profile with a rectilinear axis is described by an analytic function of the form $\delta(s) = \sqrt{1-s^2}/\rho(s)$ $(-1 \leqslant s \leqslant 1)$, where the rational function $\rho(s) > 0$. The solution [3] of the problem on perturbation of a filtration flow by a single crack in a thin layer was found to diverge for the sharp-ended crack. This points to the inefficiency of the method given in [3]. Below we obtain a general solution of the direct problem and show, that this solution is also valid in the case when the cracks are sharp-ended. A generalization of the fundamental boundary condition [1, 2] is also given. This generalization makes possible the investigation of hydromechanical interaction between the deforming cracks and the neighboring unstable filtration flow of a homogeneous liquid in a thin layer.

1. Boundary condition for the filtration flow at the crack edges. The system of functional equations of flow. Figure 1b gives a schematic representation of a transverse cross section of the macrocrack Γ in a thin oblique layer [1]. Generally speaking the crack axis is curved; below however we consider the case of a rectilinear crack $\Gamma \equiv AB$. The volume V of the element $MM'M_*M_*'$ (Fig. 1a) of the crack AB consists of the volume V_0 of cavities filled with the homogeneous liquid of the layer and the volume V_* of elastic inclusions L such as e. g. gaseous bubbles formed under the reduced pressure and moving freely in the crack during the filtration process. The position of the transverse section MM' on the axis of the crack Γ is defined by the real parameter s. The volume $V = V_0 + V_*$ may change its value for two reasons:

1) transverse sagging may take place, i. e. a narrowing (or widening) of the crack, or
2) compression (or expansion) of the elastic inclusions may take place depending on

the change in the hydromechanical pressure $p = p\,(s,\ t)$.

The transverse narrowing (sagging) of the crack Γ in the cross section $M\,(s)$ is determined by the expression

$$\Delta h = +\,\beta_0 h_0 \Delta p \qquad (\Delta V = \Delta s \Delta h) \tag{1.1}$$

where $h_0 = h_0\,(s)$ is the initial width of the crack and β_0 is the volume compressibility coefficient (the coefficient of "elastic sagging" of the crack in the cross section $M\,(s)$).

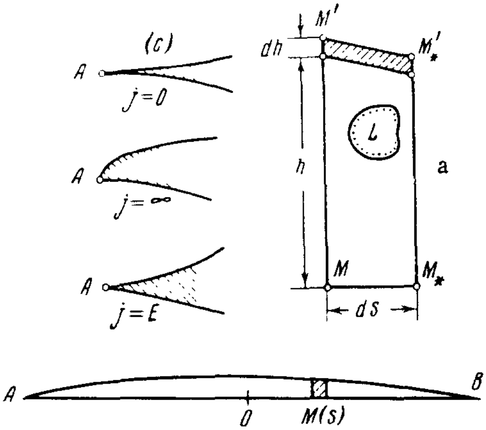

Fig. 1

Relation (1.1) is the initial relation in the theory of a beam supported on an elastic foundation [4]. The (sufficiently narrow) cavity represented by the crack Γ in the thin layer E is modelled by the elastic beam which undergoes a transverse deformation depending on the external hydromechanical pressure

$$p = p\,(s,\ t)$$

(t is the time parameter).

The change in the volume V_* of elastic inclusions L in the element $MM'M_*M_*$ is given by

$$\Delta V_* = -\,mV\beta_0 \Delta p = -\,mh\beta_* \Delta s \Delta p \tag{1.2}$$

where m is the proportion of the volume of the empty element $MM'M_*M_*'$ containing the elastic inclusions L; β_* is the volume compressibility coefficient of the elastic inclusions and $h = h\,(s,\ t)$ is the crack width at the cross section $M\,(s)$. The quantity m is analogous to porosity in the usual sense of the word, but has a more general meaning. It is therefore more correct to assign to m [5, 6] the name of crack vacuity at the cross section $M\,(s)$.

Relations (1.1), (1.2) together with the expression $\Delta V_0 = \Delta V - \Delta V$ yield

$$\Delta V_0 = (\beta_0 h_0 + m\beta_* h)\ \Delta p \Delta s \tag{1.3}$$

The volume compressibility coefficient β of the cracks is given by

$$\beta = \frac{\Delta V_0}{\Delta p V} = \beta_0 + m\beta_* \frac{h_0}{h} \qquad (V = h\Delta s,\ h_0 = h\,(s,\ 0)) \tag{1.4}$$

Using (1.4) we obtain the basic system of the functional equations for a filtration flow in a thin oblique layer [2] in the form

$$\frac{\partial}{\partial s}\left(\frac{\delta}{|\zeta'|}\frac{\partial \varphi}{\partial s}\right) - |\zeta'|\,(h_0\beta_0 + m\beta_* h)\ \frac{\mu}{k}\frac{\partial \varphi}{\partial t} = \frac{\partial \omega}{\partial s}$$

$$\varphi = \operatorname{Re} F\,(\zeta) + \operatorname{Re}\frac{1}{2\pi}\int_{(\sigma)} \frac{\omega\,(\sigma,\ t)\,\zeta'\,(\sigma)}{\zeta\,(\sigma) - \zeta\,(s)}\,d\sigma \tag{1.5}$$

Here $\zeta = \zeta\,(s)$ is a complex function of the real argument s defining the point $M\,(s)$ on the axis of the crack Γ; $F\,(\zeta)$ is the complex potential of the external flow at the point ζ; $\omega\,(s,\ t)$ is the discharge function of the liquid contained in the layer measured across the cross section M (the discharge is relative to the unit capacity of the layer), $\delta = (k_0/k)\, h$ is the effective crack width at the point M; k_0, k are the permeabilities of the crack filler and of the porous medium respectively and μ is the viscosity of the liquid in the layer. The dimension of each quantity appearing in (1.5) can be found by the methods used in [1].

For a rectilinear crack we have $\zeta = z_0 s$ $(z_0 = be^{i\beta}$, $\beta = 0$ is the polar angle, $-1 \leqslant s \leqslant 1)$. In the case of a progressive flow we have $F(\zeta) = V\zeta$ where V is the flow velocity along the real axis.

The element $MM'M_*M_*'$ of the crack Γ represents a narrow gap between two cylindrical walls MM_* and $M'M_*'$ the distance between which varies within narrow limits. Therefore it can be stated with sufficient accuracy that the longitudinal permeability of such a gap has, in the case of a laminar viscous flow, an upper limit determined by the Boussinesq formula [5] for a plane gap

$$\max k_0 = {}^1/_{12}h^2 \qquad (0 < k_0 \leqslant \max k_0) \tag{1.6}$$

The system (1.5) can be reduced to its dimensionless form with the help of such parameters as the time T, the pressure p_0, the axial crack length $2b$, the greatest width of the crack or its width H in the middle cross section, and of the following obvious relations

$$\begin{gathered} t = T\tau,\ \ \varphi = -\frac{k}{\mu}p_0P,\ \ p = p_0P,\ \ \omega = -\frac{k}{\mu}p_0\Omega,\ \ h = Hf,\ \ h_\infty = Hf_\infty \\ h_0 = Hf_0 \\ P = P(s,\tau),\ \Omega = \Omega(s,\tau),\ P_\infty = P(s,\infty),\ \Omega_\infty = \Omega(s,\infty) \\ f = f(s,\tau),\ f_0 = f(s,0),\ f_\infty = f(s,\infty),\ F = P - P_\infty,\ G = \Omega - \Omega_\infty \\ \lambda = \frac{kT}{\mu bH\beta_0},\quad \varkappa = \frac{TH^2}{12\mu\beta_0 b^2},\quad \varepsilon = m\frac{\beta_*}{\beta_0},\quad \nu = \frac{Vb\mu}{kp_0},\quad \Theta = \beta_0 p_0,\quad \frac{\varkappa}{\lambda} = \frac{H^2}{12kb} \\ f(\pm 1,\tau) = 0,\qquad \Omega(\pm 1,\tau) = 0,\qquad f(0,\tau) = 1 \end{gathered} \tag{1.7}$$

The last line in (1.7) is equivalent to the boundary conditions for the functions $h(s,\tau)$ and $\omega(s,\tau)$. Using the relations (1.7) to trasform (1.5) we obtain the following system of functional equations

$$\begin{gathered} \varkappa\frac{\partial}{\partial s}\left(f^3\frac{\partial P}{\partial s}\right) - (\varepsilon f + f_0)\frac{\partial P}{\partial \tau} = \lambda\frac{\partial\Omega}{\partial s} \\ P + \nu s = \frac{1}{2\pi}\int_{-1}^{1}\frac{\Omega(\sigma,\tau)}{\sigma - s}\,d\sigma \end{gathered} \tag{1.8}$$

In the steady state problem we have $-P(s,\infty) = P_\infty = \text{const}(\tau)$ and Eqs. (1.8) then yield

$$\begin{gathered} \varkappa f_\infty^3\frac{\partial P_\infty}{\partial s} = \lambda\Omega_\infty \\ P_\infty + \nu s = \frac{1}{2\pi}\int_{-1}^{1}\frac{\Omega_\infty(\sigma)}{\sigma - s}\,d\sigma \ \text{ when } \left\{\begin{array}{l} \Omega(s,\infty) = \Omega_\infty \\ f(s,\infty) = f_\infty \end{array}\right. \end{gathered} \tag{1.9}$$

Eliminating the function $P_\infty(s)$ from (1.9) with the accuracy allowed by the notation used, we obtain an integrodifferential equation already found in [1]

$$\frac{\lambda}{\varkappa}\frac{\Omega_\infty(s)}{f_\infty{}^3(s)}+\nu=\frac{d}{ds}\frac{1}{2\pi}\int_{-1}^{1}\frac{\Omega_\infty(\sigma)}{\sigma-s}d\sigma \tag{1.10}$$

In the direct problem a specified expression for $\Omega_\infty(s)$ is used to obtain the crack profile determined by the function $f_\infty(s)$. In the converse problem the function $f_\infty(s)$ is specified and the discharge function $\Omega_\infty(s)$ is to be determined. Certain additional smoothness requirements imposed on the functions $f_\infty(s)$ and $\Omega_\infty(s)$ reduce (1.10) to the Prandtl equation [7–10]. The solution of the converse problem however, obtained in this manner [3] has a number of substantial shortcomings. It loses its meaning when the crack is sharp-ended and this is obvious from the fact that certain improper integrals diverge. For example, for the sharply ending cracks the expression for the auxiliary function $\theta(s)$ [3] becomes infinite as $|s|\to 1$.

2. Solution of the steady state problem on the interaction of a macrocrack in a thin layer with the neighboring filtration flow using trigonometric expansions. It can easily be shown that the functional trasformation [1, 10]

$$\int_{0}^{1}\frac{\cos\pi n\sigma}{\cos\pi\sigma-\cos\pi s}d\sigma=\frac{\sin\pi ns}{\sin\pi s}\qquad\begin{pmatrix}n=0,\,1,\,2,\,\ldots;\\ -1<s<1\end{pmatrix} \tag{2.1}$$

is equivalent to the following system of conjugate transformations

$$\frac{1}{\pi}\int_{-1/2\pi}^{1/2\pi}\frac{\cos n\chi}{\sin\chi-\sin\theta}d\chi=-\frac{\sin n\theta}{\cos\theta}\qquad\begin{pmatrix}n=0,\,2,\,4,\,\ldots;\\ -1/2\,\pi<\theta<1/2\,\pi\end{pmatrix}$$

$$\frac{1}{\pi}\int_{-1/2\pi}^{1/2\pi}\frac{\sin n\chi}{\sin\chi-\sin\theta}d\chi=\frac{\cos n\theta}{\cos\theta}\qquad\begin{pmatrix}n=1,\,3,\,5,\,\ldots;\\ -1/2\,\pi<\theta<1/2\,\pi\end{pmatrix} \tag{2.2}$$

The system (1.9) can be transformed to new arguments θ and χ using the substitutions

$$f_\infty=f_\infty(s)=f_*(\theta),\ P_\infty=P_\infty(s)=P_*(\theta),\ \Omega_\infty=\Omega_\infty(s)=\Omega_*(\theta)$$
$$\Omega_\infty(\sigma)=\Omega_*(\chi),\ s=\sin\theta,\ \sigma=\sin\chi \tag{2.3}$$
$$(-1/2\,\pi\leqslant\theta\leqslant 1/2\,\pi)\quad(-1/2\,\pi\leqslant\chi\leqslant 1/2\,\pi)$$

As the result we obtain

$$\varkappa f_*{}^3(\theta)\frac{\partial P_*(\theta)}{\partial\theta}=\lambda\Omega_*(\theta)\cos\theta\qquad(-1/2\,\pi<\theta<1/2\,\pi)$$

$$P_*(\theta)+\nu\sin\theta=\frac{1}{2\pi}\int_{-1/2\pi}^{1/2\pi}\frac{\Omega_*(\chi)\cos\chi}{\sin\chi-\sin\theta}d\chi \tag{2.4}$$

Let the even function $\Omega_*(\theta)$ $(-1/2\,\pi\leqslant\theta\leqslant 1/2\,\pi)$ be sufficiently smooth (the smoothness of a function is determined by the order r of its highest continuous derivative) and represented by a uniformly converging trigonometric series in cosines of odd multiples of the argument of the form

$$\Omega_*(\theta)=\sum_{(n)}c_n\cos n\theta\qquad(n=1,\,3,\,5,\,\ldots;\ -1/2\,\pi\leqslant\theta\leqslant 1/2\,\pi) \tag{2.5}$$

Obviously the function $\Omega_*(\theta)$ satisfies the boundary condition

$\Omega_\infty(\pm 1) = \Omega_*(\pm {}^1/_2\,\pi) = 0$. Substitution of the expansion for $\Omega_*(\theta)$ given in (2.5) into the integrand in (2.4) makes possible the computation of the latter by integrating the resulting series term by term. Elementary trigonometric transformations and the use of (2.2) together yield the following expression for the pressure function $P_*(\theta)$:

$$P_*(\theta) = -\nu\sin\theta - \frac{1}{2}\sum_{(n)} c_n \sin n\theta \qquad \left(\begin{array}{c} n = 1, 3, 5; \ldots \\ -{}^1/_2\,\pi < \theta < {}^1/_2\,\pi \end{array}\right) \tag{2.6}$$

Insertion of $P_*(\theta)$ into the first equation of (2.4) now yields

$$\frac{\varkappa}{\lambda} f_*^{\,3}(\theta) = -\cos\theta \sum_{(n)} c_n \cos n\theta \left(\nu\cos\theta + \frac{1}{2}\sum_{(n)} nc_n \cos n\theta\right)^{-1}$$

$$\left(\begin{array}{c} n = 1, 3, \ldots 5; \\ -{}^1/_2\,\pi < \theta < {}^1/_2\,\pi \end{array}\right) \tag{2.7}$$

The system (2.5)–(2.7) together with the expression $s = \sin\theta$ taken from (2.3) represents the required solution of the problem in the parametric form. Here it must be remembered that the functional properties of the functions $P_*(\theta)$ (2.6) and $f_*(\theta)$ (2.7) depend on the analogous properties of the initial function $\Omega_*(\theta)$ (2.5).

Keeping the above remark in mind and depending on which of the three functions $\Omega_*(\theta)$, $P_*(\theta)$ or $f_*(\theta)$ is specified, we define the coefficients c_n ($n = 1, 3, 5, \ldots$) from the corresponding expansions (2.5), (2.6) or (2.7) by familiar methods, e.g. for an approximate solution of the problem we use the expansion in the discrete values of the argument θ over the segment $-{}^1/_2\,\pi < \theta < {}^1/_2\,\pi$. We note an elementary case analogous to the example given in [3]. Assume that $c_1 \neq 0$, $c_3 = c_5 = \ldots = 0$. Then the relations (2.5)–(2.7) give

$$\Omega_*(\theta) = c_1\cos\theta,\ P_*(\theta) = -(\nu + {}^1/_2\,c_1)\sin\theta,\ \frac{\varkappa}{\lambda} f_*^3(\theta) = -\frac{c_1}{\nu + {}^1/_2\,c_1}\cos\theta \tag{2.8}$$

at the point $\theta = 0$, $f_\infty(0) = f_*(0) = 1$, and this leads to the following equation for c_1:

$$c_1 = -\frac{\nu\varkappa}{\lambda + {}^1/_2\,\varkappa} \tag{2.9}$$

Using (2.3) we obtain explicit expressions for the functions

$$f_\infty = (1 - s^2)^{1/6},\ \ \Omega_\infty(s) = -\frac{\nu\varkappa}{\lambda + {}^1/_2\,\varkappa}(1 - s^2)^{1/2},\ \ P_\infty(s) = -\frac{\lambda\nu}{\lambda + {}^1/_2\,\varkappa}\, s \tag{2.10}$$

$$(-1 \leqslant s \leqslant 1)$$

Employing (1.7) we now obtain the solution of the problem in the dimensional quantities

$$h_\infty = H(1 - s^2)^{1/6} \tag{2.11}$$

$$\omega_\infty = \frac{2VbH^3}{H^3 + 24\,kb}(1 - s^2)^{1/2},\ \ p_\infty = -\frac{24\,Vb^2\mu}{H^3 + 24\,kb}\, s$$

$$x = bs \qquad (-1 \leqslant s \leqslant 1)$$

where x represents the abscissa of the point $M(s)$. When $x = \pm b$ we have $dh/dx = \mp\infty$, therefore the profile of the crack Γ intersects with the axis AB at the right angles. In other words, the edges of the crack $\Gamma = AB$ are not closed at the end points A and B, i.e. the crack ends are not cuspidal.

Next we consider the problem of existence of the sharp-ended cracks.

3. Conditions of existence of the sharp-ended cracks. The profile of the crack Γ can be obtained near each of its end points A and B in the parametric form using $\alpha > 0$ as the argument, from (2.7) in which the substitution $\theta = \pm (^1/_2\pi - \alpha)$ has been made. This yields

$$s = \pm\cos\theta, \; \frac{\varkappa}{\lambda} f_*{}^3 = -\sin\alpha \sum_{(n)} (-1)^{1/2(n-1)} c_n \sin n\alpha \times \qquad \left(\begin{matrix} 0 < \alpha < {}^1/_2\pi \\ n = 1, 3, 5, \ldots \end{matrix}\right) \tag{3.1}$$

$$\times \left[\nu \sin\alpha + \frac{1}{2} \sum_{(n)} (-1)^{1/2(n-1)} n c_n \sin n\alpha\right]^{-1}$$

If the numerical series $\sum\limits_{(n)} n^r c_n$ converges absolutely for $\Omega_* (\theta)$ (2.5) for $r = 7$, it also converges absolutely when $1 \leqslant r \leqslant 6$. If in addition

$$\nu + \sum_{(n)} (-1)^{1/2(n-1)} n^2 c_n \neq 0 \tag{3.2}$$

also holds, the expression (3.1) yields a limiting relation $\lim f_* = f_*(\pm {}^1/_2\pi) = 0$ for $\alpha \to 0$. This means that the boundary condition $f_\infty (\pm 1) = 0$ (1.7) holds for the function $f_\infty (s)$.

Three basic types of the crack profile form (Fig. 1c) depending on the degree (r) of smoothness of the function f_* (3.1) can be recognized:

1. sharp-ended crack if $j = dh/dx \to \mp 0$ as $x \to \pm b$,
2. blunt-ended crack if $j = dh/dx \to \pm \infty$ as $x \to \pm b$ and
3. angular crack if $j = dh/dx \to E$ ($|E| < \infty$) as $x \to \pm b$.

The proposed classification is also applicable to the asymmetric cracks; in this case however each crack end must be considered independently and matched against one of the crack types listed above.

The form of the crack AB depends on the behavior of the function dh/dx as $\alpha \to 0$, and we have

$$\frac{dh}{dx} = \pm \frac{H}{b} \frac{1}{\sin\alpha} \frac{df_*}{d\alpha} \tag{3.3}$$

Using (3.2) we obtain from (3.1) the following expansion:

$$f_*{}^3 = M\left[\alpha \sum_{(n)} \gamma_n n c_n - \frac{\alpha^3}{3!} \sum_{(n)} \gamma_n n^3 c_n + \frac{\alpha^5}{5!} \sum_{(n)} \gamma_n n^5 c_n - \right.$$

$$\left. - \frac{\alpha^7}{7!} \sum_{(n)} \gamma_n n^7 c_n\right] + O(\alpha^9), \; M = \text{const}, \; \gamma_n = (-1)^{1/2(n-1)} \tag{3.4}$$

The following theorem holds:

1) If the equations $$\sum_{(n)} \gamma_n n^r c_n = 0 \qquad (r = 1, 3, 5) \tag{3.5}$$

and the expansion (3.4) are both satisfied, then the crack Γ is sharp-ended at A and B and its form is of the type 1;

2) if at least one of equations (3.5) does not hold together with the expansion (3.4), then the crack is blunt-ended at A and B, i.e. the profile of the crack at these points is inclined at a right angle to the crack axis, and in this case the form of the crack Γ is of the type 2.

The proof is obvious and based on considering the order of smallness of the expressions (3.3) and (3.4) at $\alpha \to 0$.

The left-hand side of (3.1) is obtained under the assumption that the effective crack width $\delta = (k_0/k)\, h$ is determined by

$$\delta\,(\max k_0/k)\, h = {}^1/_{12}\,(h^3/k)$$

When the permeability k_0 is expressed in this manner, the possibility of existence of cracks of the type 3 characterized by the discharge function $\Omega_*\,(\theta)$ in its expanded form (2.5) is excluded, i.e. no cracks with angular profiles at the ends A and B exist.

Apart from the condition $f_*\,(\pm\, {}^1/_2\,\pi) = 0$ the function f_* (2.7) must also satisfy the scale condition $f_*\,(0) = 1$, therefore from (2.7) we obtain

$$\nu + \sum_{(n)} \left(\frac{\lambda}{\varkappa} + \frac{1}{2}\,n\right) c_n = 0 \qquad (n = 1,\ 3,\ 5, \ldots) \tag{3.6}$$

Thus for the sharp-ended cracks we have the system (3.5) and (3.6). The minimum number of nonzero coefficients c_n $(n = 1,\ 3,\ 5,\ \ldots)$ sufficient for constructing the profile of a sharp-ended crack is four, i.e. c_1, c_3, c_5 and c_7. The latter can be obtained from (3.5) and (3.6) by setting $c_9 = c_n = \ldots = 0$

$$c_1 - 3^r c_3 + 5^r c_5 - 7^r c_7 = 0 \qquad (r = 1,\ 3,\ 5) \tag{3.7}$$

$$\nu + \left(\frac{\lambda}{\varkappa} + \frac{1}{2}\right) c_1 + \left(\frac{\lambda}{\varkappa} + \frac{3}{2}\right) c_3 + \left(\frac{\lambda}{\varkappa} + \frac{5}{2}\right) c_5 + \left(\frac{\lambda}{\varkappa} + \frac{7}{2}\right) c_7 = 0$$

This system of equations has the following unique solution

$$c_1 = 35\rho,\ \ c_3 = 21\rho,\ \ c_5 = 7\,\rho,\ \ c_7 = \rho = -\,\varkappa\nu\,(64\,\lambda + 70\varkappa)^{-1} \tag{3.8}$$

In this case the solution of the sharp-ended crack is obtained in a finite form from (2.5)–(2.7) and (3.8). The formula 1.3.2.3.6 in [11] gives the following expression for the sum

$$\sum_{(n)} c_n \cos n\theta = 64\, c_7 \cos^7 \theta \tag{3.9}$$

4. Complex potential of the external filtration flow. The filtration flow outside the crack Γ in a thin layer is described by its complex potential $w\,(z)$ [1, 2] in the form

$$w\,(z) = Vz + \frac{1}{2\pi} \int_{-1}^{1} \frac{\omega\,(s)\, b}{bs - z}\, ds = Vz - \frac{kp_0}{\mu}\frac{1}{2\pi} \int_{-{}^1/_2\pi}^{{}^1/_2\pi} \frac{\cos\theta\, \Omega_*(\theta)}{\sin\theta - z/b}\, d\theta \tag{4.1}$$

Here z is the complex coordinate of the point $M\,(z)$ belonging to the thin layer E. Passing in the integral appearing in (4.1) to the complex argument $\tau = e^{i\theta}$ we obtain

$$w\,(z) = Vb\zeta - \frac{kp_0}{\mu}\frac{1}{8\pi i} \oint_{|\tau|=1} K\,(\tau,\zeta) \sum_{(n)} \gamma_n c_n\,(\tau^n - \tau^{-n})\, d\tau \tag{4.2}$$

$$\zeta = z/b,\ \ K\,(\tau,\zeta) = (\tau - \tau^{-1})\,(\tau^2 - 2\zeta\tau + 1)^{-1},\ \ \gamma_n = (-1)^{{}^1/_2\,(n-1)}$$

The contour integral in (4.2) is calculated using the theory of residues [1]

$$w\,(z) = Vb\zeta - \frac{kp_0}{2\mu} \sum_{(n)} (-1)^{{}^1/_2\,(n-1)}\, c_n\,(\zeta - \sqrt{\zeta^2 - 1}\,)^n \quad (\zeta = z/b;\ n = 1,\ 3,\ 5, \ldots) \tag{4.3}$$

The complex velocity $w'\,(z) = u - iv$ at the point $M\,(z)$ is obtained by differentiating the complex potential $w\,(z)$ (4.3) with respect to its argument $z = x + iy$

$$w'\,(z) = V + \frac{kp_0}{2\mu b\,\sqrt{\zeta^2 - 1}} \sum_{(n)} (-1)^{{}^1/_2\,(n-1)}\, nc_n\,(\zeta - \sqrt{\zeta^2 - 1})^n \tag{4.4}$$

$$(\zeta = z/b;\ n = 1,\ 3,\ 5, \ldots)$$

We shall show that for the type 1 cracks (sharp-ended cracks) the complex velocity at the crack ends, i. e. when $z \to \pm b$ ($\zeta \to \pm 1$) has always a bounded value. Indeed, for the type 1 cracks the first equation of (3. 5) holds, therefore (4. 4) can be represented by the following equivalent expression:

$$w'(z) = V + \frac{kp_0}{2\mu b}\sum_{(n)} (-1)^{1/2(n-1)} c_n \frac{(\zeta - \sqrt{\zeta^2-1})^n - \zeta}{\sqrt{\zeta^2-1}}$$

$$(\zeta = z/b;\ n = 1, 3, 5, \ldots) \qquad (4.5)$$

The passage to the limit in the right-hand side of (4. 4) as $\zeta \to \pm 1$ is performed by applying the l'Hospital rule to each term under the summation sign. This gives a finite value to the rate of filtration at the points $z = \pm b$

$$w'(\pm b) = V - \frac{kp_0}{2\mu b}\sum_{(n)} (-1)^{1/2(n-1)} n^2 c_n \quad (n = 1, 3, 5, \ldots) \qquad (4.6)$$

The rate of filtration at the upper and the lower crack edges in the middle cross section O is found from (4. 4) for $\zeta = \pm 0$

$$w'(\pm 0) = V - \frac{kp_0}{2\mu b}\sum_{(n)} nc_n \qquad (4.7)$$

The expressions (4. 6) and (4. 7) can only assume real values, and from this we obtain the longitudinal and transverse components of the filtration rate of the external flow at the points A, B and O (Fig. 1)

$$u_A = u_B = V - \frac{kp_0}{2\mu b}\sum_{(n)} (-1)^{1/2(n-1)} n^2 c_n, \quad v_A = v_B = 0$$

$$u_O^+ = u_O^- = V - \frac{kp_0}{2\mu b}\sum_{(n)} nc_n, \quad v_O^+ = v_O^- = 0 \qquad (4.8)$$

Using (3. 8) and (4. 8) we obtain the following expressions for the example from Sect. 3:

$$u_A = u_B = \frac{32\lambda + 28\varkappa}{32\lambda + 35\varkappa} V, \qquad v_A = v_B = 0$$

$$u_O^+ = u_O^- = \frac{32\lambda + 70\varkappa}{32\lambda + 35\varkappa} V, \qquad v_O^+ = v_O^- = 0 \qquad (4.9)$$

Here we note that (4. 6) is valid for the cracks of type 2 only if the first equation of (3. 5) holds, otherwise the filtration rate of the external flow increases in the neighborhood of each end of the blunt-ended crack without bounds. An approximate sketch of the streamlines near the cracks of all types is given in Fig. 2.

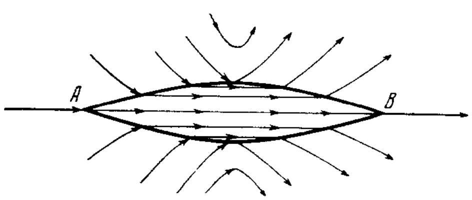

Fig. 2

5. Approximate theory of deformation of a crack when the longitudinal filtration flow is unsteady. The preliminary solution of the problem for the case of a steady flow when $\tau \to \infty$ is assumed known and given in the form of (2. 5)–(2. 7) obtained from (1. 9). On the basis of (1. 7) we can transform (1. 1) as follows:

$$f(s,\ \tau) = f_0\,[1 + \Theta\,(F - F_0)], \qquad F = P\,(s,\tau) - P\,(s,\infty)$$
$$F_0 = P\,(s,0) - P\,(s,\infty), \qquad f_0 = f\,(s,0) \tag{5.1}$$

from which for $\tau \to \infty$ we have

$$f_\infty = f_0\,(1 - \Theta F_0) \tag{5.2}$$

The relation (5.2) connects the final f_∞ and initial f_0 profiles of the crack AB. Eliminating f_0 from (5.1) and (5.2) we obtain

$$\frac{f\,(s,\tau)}{f_\infty} = 1 + \frac{\Theta F}{1 - \Theta F_0} \tag{5.3}$$

Numerical calculations show that the coefficient of deformation Θ is sufficiently small, consequently the basic system of equations (1.8) can be reduced to the following functional equations

$$\varkappa\,\frac{\partial}{\partial s}\left(f_\infty^3\,\frac{\partial F}{\partial s}\right) - (\varepsilon + 1)\,f_\infty\,\frac{\partial F}{\partial \tau} = \lambda\,\frac{\partial G}{\partial s}\,, \quad F = \frac{1}{2\pi}\int\limits_{-1}^{1}\frac{G\,(\sigma\tau)}{\sigma - s}\,d\sigma$$
$$(F = P - P_\infty,\ \ G = \Omega - \Omega_\infty) \tag{5.4}$$

The first equation of (5.4) approximates the first equation of (1.8) with the accuracy of up to the terms of the $O\,(\Theta)$-order. The second equation of (5.4) is obtained by eliminating the term vs from (1.8) and (1.9). Using the substitutions

$$F = F\,(s,\tau) = F_*\,(\theta,\tau), \qquad G = G\,(s,\tau) = G_*\,(\theta,\tau)$$
$$s = \sin\theta\ (-{}^1/_2\,\pi < \theta < {}^1/_2\,\pi),\ \sigma = \sin\chi\ \ (-{}^1/_2\pi < \chi < {}^1/_2\,\pi)$$
$$F_{*0} = F_*\,(\theta,0), \qquad G_{*0} = G_*\,(\theta,0)$$
$$f = f\,(\theta,\tau),\ f_* = f\,(\theta,\infty), \quad f_0 = f\,(\theta,0) \tag{5.5}$$

we can write (5.4) with θ and χ as the arguments. After some manipulations (5.3) reduces to

$$\frac{f}{f_*} = 1 + \frac{\Theta F_*}{1 - \Theta F_{*0}} \tag{5.6}$$

and (5.4) is replaced by

$$\varkappa\,\frac{\partial}{\partial\theta}\left(\frac{f_*^3}{\cos\theta}\,\frac{\partial F_*}{\partial\theta}\right) - (\varepsilon + 1)\,f_*\cos\theta\,\frac{\partial F_*}{\partial\tau} = \lambda\,\frac{\partial G_*}{\partial\theta}$$
$$F_* = \frac{1}{2\pi}\int\limits_{-{}^1/_2\pi}^{{}^1/_2\pi}\frac{G_*\,(\chi,\tau)\cos\chi}{\sin\chi - \sin\theta}\,d\chi \tag{5.7}$$

where the function $f_*\,(\theta)$ is known from (2.7).

The solution of the functional equations (5.7) is sought in the form of trigonometric expansions in odd harmonics, using the methods given in Sect. 4. Instead of the expansion (2.5) we take

$$G_*\,(\theta,\tau) = \sum_{(n)} g_n\,(\tau)\cos n\theta \qquad (n = 1, 3, 5, \ldots) \tag{5.8}$$

Using transformations (2.2), the second equation of (5.7) is represented in the form

$$F_*\,(\theta,\tau) = -\frac{1}{2}\sum_{(n)} g_n\,(\tau)\sin n\theta \qquad (n = 1, 3, 5, \ldots) \tag{5.9}$$

Inserting the function $G_*\,(\theta,\tau)$ (5.8) and $F_*\,(\theta,\tau)$ (5.9) into the first functional equation of (5.7) we reduce the problem to an infinite system of the linear differential equations for $g_n(\tau)$. Another problem may be posed here that of finding the eigenvalues of this system.

We shall begin the preliminary approximate analysis of (5.7) by considering the solution of the steady state problem (2.5)–(2.7) obtained in the form

$$\Omega_*(\theta) = -\nu\Lambda(3\cos\theta + \cos 3\theta) = -4\nu\Lambda\cos^3\theta \quad \left(\begin{array}{c} -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \\ \Lambda = \dfrac{\varkappa}{3\varkappa + 8\lambda} \end{array}\right)$$
$$P_*(\theta) = -\nu\sin\theta + \tfrac{1}{2}\nu\Lambda(3\sin\theta + \sin 3\theta)$$
$$f_*^3 = \frac{8\lambda}{3\varkappa + 8\lambda}\cos^3\theta\,(1 - 3\Lambda\cos^3\theta\sec\theta)^{-1}, \quad s = \sin\theta \tag{5.10}$$

To simplify the analysis we assume that the quantities $\varkappa$ and λ satisfy the inequality

$$3\Lambda \ll 1 \tag{5.11}$$

Then the last equation of (5.10) gives the following approximate expression

$$f_* = l\cos\theta, \qquad l = [8\lambda/(3\varkappa + 8\lambda)]^{1/3} \tag{5.12}$$

Obviously, the corresponding case is that of the blunt-ended crack AB.

In the expansions for G_* (5.8) and F_* (5.9) we assume the following:

$$g_n(\tau) = e^{-r\tau} g_n, \qquad g_n = \mathrm{const}\,(\tau)$$

The first equation of (5.7) leads, after the substitution of (5.8), (5.9), (5.12), to the functional series

$$-\frac{\varkappa}{8\lambda} l^3 \sum_{(n)} n g_n [2n\sin n\theta + (n-2)\sin(n-2)\theta + (n+2)\sin(n+2)\theta] +$$
$$+ \frac{\varepsilon+1}{8\lambda} lr \sum_{(n)} g_n [2\sin n\theta + \sin(n-2)\theta + \sin(n+2)\theta] =$$
$$= \sum_{(n)} n g_n \sin n\theta \qquad (n = 1, 3, 5, \ldots) \tag{5.13}$$

The latter holds for g_1 and $g_3 \neq 0$ and $g_5 = g_7 = \ldots = 0$, therefore the conditions of solvability of Eq. (5.13) are

$$r = \frac{15\varkappa}{\varepsilon+1} l^2, \quad \frac{\varkappa}{\lambda} l^3 = \frac{1}{2}, \quad g_1 = 3g_3 \tag{5.14}$$

Insertion of the expression for l from (5.12) into the second equation of (5.14) gives the value $\lambda/\varkappa = {}^{13}/_{8}$, therefore we have $l^3 = {}^{13}/_{16}$. The condition (5.11) holds, consequently the approximate solution of the problem has a meaning and we obtain the following expressions for G_* and F_* when $A = {}^{1}/_{2}\, g_3$:

$$G_* = 2A\,e^{-r\tau}(3\cos\theta + \cos 3\theta), \quad G_{*0} = 2A(3\cos\theta + \cos 3\theta)$$
$$F_* = -A\,e^{-r\tau}(3\sin\theta + \sin 3\theta), \quad F_{*0} = -A(3\sin\theta + \sin 3\theta) \tag{5.15}$$

From (5.6), (5.12) and (5.15) we obtain the required solution of the problem in the form

$$P = P_* + F_*, \quad \Omega = \Omega_* + G_*, \quad s = \sin\theta \quad (-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi)$$
$$P_* = -\nu\sin\theta + \frac{1}{2}\,\frac{\varkappa\nu}{3\varkappa + 4\lambda}(3\sin\theta + \sin 3\theta),$$

$$\Omega_* = -\frac{\varkappa\nu}{3\varkappa + 4\lambda}(3\cos\theta + \cos 3\theta)$$

$$f = l\cos\theta\left[1 - \frac{A\theta(3\sin\theta + \sin 3\theta)}{1 + A\theta(3\sin\theta + \sin 3\theta)}e^{-r\tau}\right] \qquad (5.16)$$

The constant A is obtained from the total variation in the pressure drop ΔP over the whole crack length AB over the time by which the steady state is established, i. e. over the variation of τ from $\tau = 0$ to $\tau = \infty$. The present value of the total pressure drop between the points A and B (Fig. 3) is written in the form

$$\Delta P = \Delta P_\infty + 4Ae^{-r\tau}$$

$$(4A = \Delta P_0 - \Delta P_\infty) \qquad (5.17)$$

From (5.16) we obtain

$$\omega(s) = -\frac{k}{\mu}p_0\Omega = \frac{k}{\mu}p_0\left(\frac{\varkappa\nu}{3\varkappa + 4\lambda} - 2Ae^{-r\tau}\right)(3\cos\theta + \cos 3\theta)$$

$$s = \sin\theta \qquad (-{}^1/_2\pi < \theta < {}^1/_2\pi) \qquad (5.18)$$

Inserting the expression for the discharge function ω (5.18) into the expression (4.1)–(4.3) for $w(z)$ gives

$$w(z) = Vb\zeta - \frac{kp_0}{2\mu}\left(\frac{\varkappa\nu}{3\varkappa + 4\lambda} - 2Ae^{-r\tau}\right)\left[3\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \right.$$

$$\left. - \left(\zeta - \sqrt{\zeta^2 - 1}\right)^3\right] \qquad (\zeta = z/b) \qquad (5.19)$$

The complex potential $w(z)$ (5.9) describes the external velocity field and the argument τ is not given explicitly in the expression for $w(\ldots)$. Obviously the given filtration field tends asymptotically to a steady state as $\tau \to \infty$.

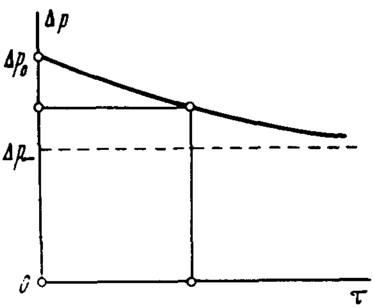

Fig. 3

Figs. 4 and 5 depict the initial (S_0), the present (S) and the final (S_*) crack profile and the corresponding plots (R_0, R, R_*) of the pressure along the crack AB. Fig. 5 shows clearly that the initial asymmetry of the crack AB vanishes asymptotically and the crack profile becomes symmetric (elliptical) with respect to the middle cross section when $\tau \to \infty$. Since at the same time the slope of the pressure curve (R) decreases, the left-hand side of the crack is found to be under a smaller load than the right-hand side. The hydrodynamic pressure in the right half of the crack increases with time, hence the profile of the right half of the crack OB moves upward with time, while the profile of the left half of the crack AO is displaced downwards relative to the steady state (elliptical) crack profile. In this manner, the initial (pear-shaped) crack profile degenerates into the final (elliptical) profile. Obviously, all this describes correctly, from the qualitative point of view, the process of deformation of the crack profile during the filtration of the layer liquid along the crack.

The quantitative detailed analysis of the crack deformation depending on the filtration effects and based on the theory proposed here, is beyond the scope of this paper. The behavior of the sharp-ended cracks can be investigated in the same manner. This will however require more complicated initial expressions (5.10) and (5.12). Otherwise

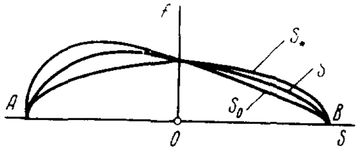

Fig. 4

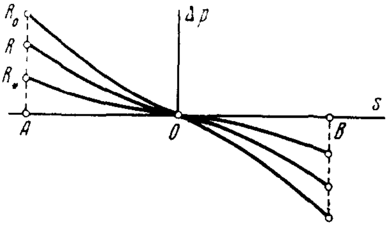

Fig. 5

the approach employed in obtaining the solution to the problem here remains valid.

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